

**Math 1B Midterm 1 Review Answers**

[1] [a] By Net Change Theorem,

$$s(13) - s(1) = \int_1^{13} s'(t) dt = \int_1^{13} v(t) dt \Rightarrow s(13) = s(1) + \int_1^{13} v(t) dt = 21 + \int_1^{13} v(t) dt \text{ feet}$$

[b] [i]  $21 + (12 + 18 + 17) * 4 = 209$  feet

[ii]  $21 + (14 + 24 + 8) * 4 = 205$  feet

[iii]  $21 + (18 + 17 + 5) * 4 = 181$  feet

[2]  $\int_{\pi}^{\frac{3\pi}{2}} 2 \sin x dx = -2$  or  $\int_{2\pi}^{3\pi} \sin \frac{1}{2} x dx = -2$

$$\begin{aligned} [3] \quad & \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left[ \left( -1 + \frac{3i}{n} \right)^2 + 3 \left( -1 + \frac{3i}{n} \right) + 2 \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left( \frac{3i}{n} + \frac{9i^2}{n^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left( \frac{3}{n} \sum_{i=1}^n i + \frac{9}{n^2} \sum_{i=1}^n i^2 \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left( \frac{3}{n} \frac{n(n+1)}{2} + \frac{9}{n^2} \frac{n(n+1)(2n+1)}{6} \right) \\ &= \lim_{n \rightarrow \infty} 3 \left( \frac{3(n+1)}{2n} + \frac{3(n+1)(2n+1)}{2n^2} \right) \\ &= 3 \left( \frac{3}{2} + \frac{6}{2} \right) \\ &= \frac{27}{2} \end{aligned}$$

$$\begin{aligned} [4] \quad & \int_{-2}^8 x dx - \int_{-2}^8 \sqrt{25 - (x-3)^2} dx \\ &= \text{area of } 8 \times 8 \text{ triangle} - \text{area of } 2 \times 2 \text{ triangle} - \text{area of semicircle of radius } 5 \text{ (with center at } (3, 0)) \\ &= \frac{1}{2}(8 \times 8) - \frac{1}{2}(2 \times 2) - \frac{1}{2}\pi(5)^2 = 30 - \frac{25}{2}\pi \end{aligned}$$

$$\begin{aligned} [5] \quad & \frac{1}{2} \leq \sin x \leq 1 \text{ on } \left[ \frac{\pi}{6}, \frac{\pi}{2} \right] \\ & \Rightarrow \frac{1}{2}x \leq x \sin x \leq x \text{ on } \left[ \frac{\pi}{6}, \frac{\pi}{2} \right] \\ & \Rightarrow \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}x dx \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x dx \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x dx \\ & \Rightarrow \frac{1}{4}x^2 \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x dx \leq \frac{1}{2}x^2 \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \end{aligned}$$

$$\Rightarrow \frac{1}{4} \left[ \left( \frac{\pi}{2} \right)^2 - \left( \frac{\pi}{6} \right)^2 \right] \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \leq \frac{1}{2} \left[ \left( \frac{\pi}{2} \right)^2 - \left( \frac{\pi}{6} \right)^2 \right]$$

$$\Rightarrow \frac{1}{4} \left( \frac{\pi}{6} \right)^2 (3^2 - 1^2) \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \leq \frac{1}{2} \left( \frac{\pi}{6} \right)^2 (3^2 - 1^2)$$

$$\Rightarrow \frac{1}{4} \frac{\pi^2}{36} (8) \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \leq \frac{1}{2} \frac{\pi^2}{36} (8) \quad \Rightarrow \frac{\pi^2}{18} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \leq \frac{\pi^2}{9}$$

$$[6] \quad \int_7^{-5} f(x) \, dx + \int_{-1}^{10} f(x) \, dx + \int_{10}^7 f(x) \, dx = \int_7^{-5} f(x) \, dx + \int_{-1}^7 f(x) \, dx = \int_{-1}^{-5} f(x) \, dx$$

$$[7] \quad [a] \quad g(3) = \int_{-2}^3 f(t) \, dt = \frac{1}{2}(3 \times 3) - \frac{1}{2}(2 \times 1) = \frac{7}{2} \text{ and } g'(3) = f(3) = -1$$

[b] Critical numbers of  $g$  occur where  $g'(x) = 0$  or is undefined in the domain of  $g$ .

By FTC Part 1, since  $f$  is continuous on  $[-5, 5]$ ,  $g$  is differentiable on  $(-5, 5)$ , and continuous on  $[-5, 5]$ .

So the domain of  $g$  includes  $[-5, 5]$ .

So, the critical numbers of  $g$  occur where  $g'(x) = f(x) = 0$  or is undefined in  $[-5, 5]$  ie. at  $x = -4, -2, 1$  and  $4$ .

[c]  $g$  is decreasing where  $g'(x) < 0$ , so  $g$  is decreasing on  $[-\infty, -4]$  and  $[1, 4]$ .

[d] Local minima of  $g$  occur at critical numbers of  $g$  where  $g'(x)$  changes from negative to positive, so the local minima of  $g$  are  $x = -4$  and  $4$ .

[e]  $g$  is concave up where  $g'(x)$  is increasing, so  $g$  is concave up on  $[-\infty, -3]$ ,  $[-2, 0]$  and  $[3, \infty]$ .

[f] Inflection points of  $g$  occur where  $g$  is continuous and  $g'(x)$  changes from increasing to decreasing, or vice versa, so the inflection points of  $g$  are  $x = -3, -2, 0$  and  $3$ .

$$[8] \quad g'(x) = 3x^2 \ln(1+x^6) - 2x \ln(1+x^4)$$

$$g''(x) = 6x \ln(1+x^6) + 3x^2 \frac{6x^5}{1+x^6} - \left( 2 \ln(1+x^4) + 2x \frac{4x^3}{1+x^4} \right)$$

$$g''(1) = 6 \ln 2 + 3 \left( \frac{6}{2} \right) - 2 \ln 2 - 2 \left( \frac{4}{2} \right) = 4 \ln 2 + 5$$

$$[9] \quad \frac{d}{dx} \left( 4 + \int_a^x \frac{1}{f(t)} \, dt \right) = \frac{d}{dx} 2\sqrt{x}$$

$$4 + \int_a^x \frac{1}{\sqrt{t}} \, dt = 2\sqrt{x}$$

$$\Rightarrow \frac{1}{f(x)} = \frac{1}{\sqrt{x}}$$

$$\Rightarrow 4 + 2\sqrt{t} \Big|_a^x = 2\sqrt{x}$$

$$\Rightarrow f(x) = \sqrt{x}$$

$$\Rightarrow 4 + 2\sqrt{x} - 2\sqrt{a} = 2\sqrt{x}$$

$$\Rightarrow 4 - 2\sqrt{a} = 0$$

$$\Rightarrow a = 4$$

[10] the number of pounds Morgan gained from when he was 8 years old to when he was 15 years old

[11] [a]  $v(t) - v(0) = \int_0^t v'(x) dx = \int_0^t a(x) dx \Rightarrow v(t) = v(0) + \int_0^t a(x) dx = 4 + (3x - x^2) \Big|_0^t = 4 + 3t - t^2$

$$\int_1^6 v(t) dt = \left( 4t + \frac{3}{2}t^2 - \frac{1}{3}t^3 \right) \Big|_1^6 = 4(6-1) + \frac{3}{2}(6^2 - 1^2) - \frac{1}{3}(6^3 - 1^3) = 20 + \frac{105}{2} - \frac{215}{3}$$

$$= 20 + 52\frac{1}{2} - 71\frac{2}{3} = \frac{5}{6} \text{ meters}$$

[b]  $v(t) = 4 + 3t - t^2 = (4-t)(1+t) \geq 0$  only on  $[-1, 4]$

$$\int_1^6 |v(t)| dt = \int_1^4 (4 + 3t - t^2) dt + \int_4^6 -(4 + 3t - t^2) dt = \left( 4t + \frac{3}{2}t^2 - \frac{1}{3}t^3 \right) \Big|_1^4 - \left( 4t + \frac{3}{2}t^2 - \frac{1}{3}t^3 \right) \Big|_4^6$$

$$= 4(4-1) + \frac{3}{2}(4^2 - 1^2) - \frac{1}{3}(4^3 - 1^3) - \left( 4(6-4) + \frac{3}{2}(6^2 - 4^2) - \frac{1}{3}(6^3 - 4^3) \right)$$

$$= 12 + \frac{45}{2} - 21 - \left( 8 + 30 - \frac{152}{3} \right) = 12 + 22\frac{1}{2} - 21 - 8 - 30 + 50\frac{2}{3} = 26\frac{1}{6} \text{ meters}$$

[12]  $\int_{-1}^2 (6 + 4f(2t+1)) dt = \int_{-1}^2 6 dt + 4 \int_{-1}^2 f(2t+1) dt$

Let  $u = 2t + 1$

$$\frac{du}{dt} = 2 \Rightarrow dt = \frac{1}{2} du$$

$$t = -1 \Rightarrow u = -1$$

$$t = 2 \Rightarrow u = 5$$

$$\int_{-1}^2 f(2t+1) dt = \int_{-1}^5 \frac{1}{2} f(u) du = \frac{1}{2} \int_{-1}^5 f(u) du = \frac{7}{2}$$

$$\int_{-1}^2 6 dt + 4 \int_{-1}^2 f(2t+1) dt = 6(2 - (-1)) + 4\left(\frac{7}{2}\right) = 32$$

[13]  $\int_{-6}^6 (x^3 \sqrt{2 + \cos x} - \sqrt{144 - (x+6)^2}) dx = \int_{-6}^6 x^3 \sqrt{2 + \cos x} dx - \int_{-6}^6 \sqrt{144 - (x+6)^2} dx$

$(-x)^3 \sqrt{2 + \cos(-x)} = -x^3 \sqrt{2 + \cos x} \Rightarrow$  first integrand is odd and continuous, over a symmetric interval

$$\int_{-6}^6 x^3 \sqrt{2 + \cos x} dx = 0$$

$$\int_{-6}^6 \sqrt{144 - (x+6)^2} dx = \text{area of quarter circle of radius 12 (with center at } (-6, 0)) = \frac{1}{4} \pi (12)^2 = 36\pi$$

$$\int_{-6}^6 (x^3 \sqrt{2 + \cos x} - \sqrt{144 - (x+6)^2}) dx = -36\pi$$